Reliability Analysis of Hierarchical Systems Using Statistical Moments

Gerard L. Reijns and Arjan J. C. van Gemund, *Member, IEEE*

***Abstract*—In many practical engineering circumstances, systems reliability analysis is complicated by the fact that the failure time distributions of the constituent subsystems cannot be accurately modeled by standard distributions. In this paper, we present a low-cost, compositional approach based on the use of the first four statistical moments to characterize the failure time distributions of the constituent components, subsystems, and top-level system. The approach is based on the use of Pearson distributions as an intermediate analytical vehicle, in terms of which the constituent failure time distributions are approximated. The analysis technique is presented for k-out-of-n systems with identical subsystems, series systems with different subsystems, and systems exploiting standby redundancy. The moment-in-moment-out approach allows for the analysis of systems with arbitrary hierarchy, and arbitrary (unimodal) failure time distributions, provided the subsystems are independent such that the resulting failure time can be expressed in terms of sums or order statistics. The technique consistently exhibits very good accuracy (on average, much less than 1 percent error) at very modest computing cost.**

***Index Terms*—Failure time analysis, hierarchical systems, order statistics, Pearson approximation, redundant systems.**

NOTATIONS

T	failure time of composite system
$f(x)$	failure time pdf of composite system
$F(x)$	failure time cdf of composite system
n	number of units (subsystems)
T_i	failure time of unit i
$f_i(x)$	failure time pdf of unit i
$F_i(x)$	failure time cdf of unit i
$g_i(x)$	Pearson approximation of $f_i(x)$

ACRONYMS[1]

BDD	Binary Decision Diagram
MTBF	Mean Time Between Failures

I. INTRODUCTION

SYSTEM reliability analysis often concerns analysing system compositions, such as series and parallel systems with either active (hot) redundancy or standby (cold) redundancy. Let T_i, $i = 1, \ldots, n$ denote the failure time of the constituent subsystems (units). Let T denote the failure time of the composite system, such that $T = T_1 \oplus, \ldots, \oplus T_n$, where for the above-mentioned, basic system compositions, the $\oplus$ operator equals min, max, and+, respectively. Let $f_i(x)$ denote the probability distribution function (pdf) of the failure time of unit i, and let $f(x)$ denote the pdf of the failure time of the composite system. Let $F_i(x)$, and $F(x)$ denote the cumulative distribution functions (cdf), respectively. For active redundancy, and identical units the above series & parallel system composites generalize into k-out-of-n systems. A k-out-of-n system is operational when at least k out of the n units work, where for series systems $k = n$ while for parallel systems $k = 1$. For independent units, it holds [22]

$$F(x) = \sum_{|T| \geq k} \left(\prod_{i \in T} F_i \right) \left(\prod_{i \notin T} (1 - F_i) \right) \tag{1}$$

where T is a set of indices ranging over all choices $\{i_1, \ldots i_m\}$ such that $k \leq m \leq n$, and $i_1 < i_2 < \ldots < i_m$.

Although one frequently encounters different types of units in a series system ($k = n$), this is quite unusual in a k-out-of-n system with redundant units ($k < n$). Hence, the system space we consider are k-out-of-n systems for *identical* units (from now on referred to as "k-out-of-n systems"), and series systems for *different* units (from now on referred to as "series systems", as series systems for identical units fall in the former class).

For k-out-of-n systems (identical units), (1) reduces to

$$F(x) = 1 - \sum_{i=k}^{n} \binom{n}{i} [1 - F_1(x)]^i \; F_1(x)^{n-i} \tag{2}$$

which yields [19]

$$f(x) = \frac{n!}{(k-1)!(n-k)!} F_1^{n-k}(x) \, [1 - F_1(x)]^{k-1} \, f_1(x) \tag{3}$$

For series systems (different units), (1) reduces to

$$F(x) = 1 - \prod_{i=1}^{n} (1 - F_i) \tag{4}$$

which yields

$$f(x) = \sum_{i=1}^{n} f_i(x) \prod_{j=1, j \neq i}^{n} (1 - F_j(x)) \tag{5}$$

Manuscript received July 23, 2006; revised February 25, 2007; accepted March 9, 2007. Associate Editor: M. Xie.

G. L. Reijns, retired, was with the Delft University of Technology, The Netherlands.

A. J. C. van Gemund is with the Delft University of Technology, The Netherlands.

Digital Object Identifier 10.1109/TR.2007.903132

[1]The singular and plural of an acronym are always spelled the same.

Many reliability characterizations have been studied for k-out-of-n systems, such as the general reliability function $R(x) = 1 - F(x)$, the actively parallel redundancy function, series reliability function, and the mean residual lifetime function [1], [10], in combination with exponential, geometric, Weibull, and hyper-exponential distributions of the constituent units [13], [22]. In many cases, the functions are expressed in terms of the pdf or cdf of the individual subsystems. For practical application, this requires knowledge of the distribution functions, which are typically standard distributions. For tractability reasons, it is, for instance, often assumed that the components have a negative exponential pdf. While this assumption enables a simple analysis, and use of Markov models [17], [18], the resulting failure time errors of the composite system can be significant when the components have indeed different types of distributions, in particular, when n is large.

Apart from analysis accuracy, from a system engineering point of view, a reliability analysis should also be applicable to a hierarchical system, such as a k-out-of-n system of which the constituent subsystems are k-out-of-n systems themselves, and so on. Such a compositional approach to reliability analysis is impossible when based on standard distribution functions, as for anything but Coxian phase type distributions, the resulting distribution function will not anymore be standard. In [2], an analysis is presented of a hierarchical (weighted) k-out-of-n system, which, however, is limited to a 2-stage system.

Another consideration in system engineering is analysis cost. Many studies have been performed in system optimization where the reliability of k-out-of-n systems is computed within a system cost/reliability optimization loop [21]. To allow practical application, the analysis method should pair accuracy with low cost, in particular with respect to a derived optimization metric such as the MTBF.

In this paper, we present a low-cost, compositional method to study systems composed of independent k-out-of-n systems, series systems, and systems with standby redundancy, of arbitrary hierarchy, and failure time distributions, provided the subsystems (units) are independent such that the resulting failure time can be expressed in terms of sums or order statistics. The method is based on the use of the first four statistical moments to characterize the failure time distributions of both the constituent components, subsystems, and resulting top-level system. The approach provides very accurate statistical moments of the composite system, even when dealing with many independent, identical units. Our method also enables the analysis of a system of which its components can have a mixture of different types of failure time distributions. The approach is based on the use of the Pearson approximation as an intermediate distribution vehicle to compute the $(n - k + 1)$-th order statistic by simple, numerical integration in terms of its associated statistical moments.

The motivation for this approach can be summed up as follows. First, the characterization of failure time distributions in terms of moments allows for arbitrary distributions, provided the first four moments give a necessary and sufficient set to capture standard distributions such as the Gaussian, and negative-exponential distributions with very high accuracy. Second, the moments provide a compact way of deriving related properties such as the 95th percentile of the $R(x) = 1 - F(x)$ function, the time at which the system operates with 95 percent probability. Third, the Pearson distributions provide an appropriate means to compute the resulting (four) moments of the composition given the (four) moments of the constituent parts with high accuracy, and very modest cost. This moment-in-moment-out interface allows computing arbitrary hierarchies of systems based on k-out-of-n systems, and series systems. Fourth, the moment framework allows a trivial inclusion of the analysis for standby redundancy, next to standard active redundancy, as this translates to merely computing the moments of the sum of variates in terms of their individual moments. Finally, the moment method, combined with the Pearson distributions, offers a low-cost analysis complexity. In spite of the use of numerical integration at the core of the derivation procedure, our analysis method has modest computing cost (0.040 CPU seconds on a 750 MHz AMD processor), independent of subsystem replication n.

Other authors have also studied reliability in terms of statistical moments, often in the context of system performance analysis, where, e.g., the parallel systems reliability problem translates to the execution time analysis of a composition of tasks running in parallel. For general, symmetric distributions, approximate analytical first and/or second moment solutions for parallel systems execution time have been reported [8], [9], [12]. However, most of the failure time pdf are found to be asymmetric in practice, which is not a requirement to our approach. Our approach also extends the number of moments that are involved in the analysis, which is essential to accuracy (as explained in Section II). In the case of parallel systems, the authors of [8], [11], [16] discuss analytic solutions, but restrict the analysis to either specific distributions, or assume distributions to be symmetric. In [7], the authors succeeded in solving parallel systems by first solving the binary instance, but the computing cost can become excessive. A solution for parallel and series systems is presented in [5], [6], and does not introduce unrealistic restrictions on $f_1(x)$. It approximates a component failure time distribution by a lambda distribution having four parameters, which can be determined if the input moments are known. An extension of the method is given in [4]. Unfortunately, the lambda distribution method turns out to be less accurate, and less stable in computing than the method presented in the current paper. In addition, the four Pearson characterization parameters can directly be expressed in the four given or measured moments, quite different from the method presented in [5], [6] where a table look-up or linear programming method is required to obtain these parameters. Moreover, our method enables the calculation of the failure time pdf of the composite of components of different type, unlike [5].

An alternative, probabilistic approach to computing the reliability of systems comprising independent units with arbitrary failure time distributions is based on the use of binary decision diagrams (BDD) [20], [23], which allow an efficient reliability analysis for arbitrary system compositions (fault trees). However, the approach is currently limited to the computation of the first moment only (also see Section VI).

An application of the Pearson distribution has been presented in [15] for the execution time analysis of parallel task compositions, where a very good accuracy was demonstrated (generally better than 1 percent for various distributions, including negative exponential, Gaussian, and uniform distributions). In contrast, the current paper introduces the approach into the system reliability field. It includes an extension of the mathematics to handle k-out-of-n redundancy, series systems of different units, as well as standby redundancy. Furthermore, the paper presents application examples from the aerospace domain including hierarchical systems analysis, and also describes analytic and pseudo-analytic solutions for particular combinations of negative exponential and/or Gaussian distributions, which are used as comparison to evaluate the accuracy of our technique.

The remainder of the paper is organized as follows. In the next section we provide a rationale for our choice of using the first four statistical moments. In Section III, we present our analysis technique for standby redundant systems, active redundant k-out-of-n systems, and series systems (of different units). In Section IV, we discuss numeric implementation issues. In Section V, we show applications of our technique to hierarchical example systems from the aerospace domain, including an evaluation of its numeric accuracy. We conclude the paper in Section VI.

II. Moments Characterization

As mentioned in the Introduction, our moment approach can also be seen as a generalization of the use of mean & variance in distribution characterization. Like mean & variance, the associated benefit is a low analysis complexity. Unlike mean & variance, however, our extended approach captures essential information on the pdf (as will be shown later on), at negligible cost penalty. The reason to limit our approach to the first four moments is motivated by the following. First, and foremost, the four-parameter Pearson distribution is determined by the first four moments. Furthermore, it has been shown [14] that the first four moments allows us to reconstruct the original distribution while introducing an acceptable error, and to distinguish between well-known standard distributions. Finally, in measurements, lower moments are also more robust than higher moments. As the measured values of higher moments are sensitive to sampling fluctuations, including higher moments does not always imply that precision will be improved.

Before presenting our analysis method, we provide two examples to illustrate that one needs to approximate an arbitrary failure time pdf of a component in terms of its first four moments in order to significantly increase accuracy when compared to, e.g., mean & variance only. In the first example, we consider two different series systems, each comprised of 64 non-redundant components, each component having a pdf with a mean of 4, variance of 1, and kurtosis of 6; but with a skewness of 1.70 in one set, and -1.70 in the other set. When the skewness is positive, reliability analysis (using our high-accuracy technique) results in a mean of the set's failure time of 3.13 with a variance of 0.02, while the other set exhibits a mean of 0.53 with a variance of 0.64. Thus, the third moment cannot be ignored.

The second example illustrates that also the fourth moment is not to be neglected. Consider again two different series systems, each comprised of 64 components, each component having a pdf with a mean of 3, variance of one, and skewness of -0.18; but with a kurtosis of 1.9 in the first set, and 5.00 in the other set. This results in a mean of the first composite set's failure time of 1.11 with a variance of 0.02, while the second set measures a mean of 0.25 with a variance of 0.80. Due to its accuracy, we have chosen to characterize an arbitrary failure time distribution of a basic component using the Pearson approximation, which is determined in terms of the first four moments.

III. Moments Analysis

As standby redundancy is associated with addition, the moments of the composite's failure time pdf can be analytically expressed in terms of the constituent moments. For active redundant k-out-of-n systems (identical units), and series systems with different units, the derivation of the resulting moments is based on the use of the Pearson distribution as an intermediate vehicle. We convert the constituent moments into the Pearson distributions $f_i(x)$ (and $F_i(x)$). Next, we compute the failure time pdf $f(x)$ according to (3) (k-out-of-n systems), or (5) (series system). Subsequently, we compute the four statistical moments

$$\mathsf{E}[X^r] = \int_{-\infty}^{+\infty} x^r f(x)\mathrm{d}x \tag{6}$$

for $r = 1, \ldots, 4$. In practice, the application of the Pearson distribution involves a translation of $\mathsf{E}[X]$ to zero, as well as a number of additional calculations for numerical stability, all of which will be outlined in the sequel. We start with the derivation for standby redundancy, and then proceed with the derivation for k-out-of-n systems (identical units), and series systems (non-identical units).

A. Standby Redundancy

In terms of the failure time moments, standby redundancy corresponds to computing the sum of the individual moments, in general given by [5]

$$\mathsf{E}[T^r] = \sum_{i=0}^{r} \binom{r}{i} \mathsf{E}[T_1^{r-i}]\mathsf{E}[T_2^i].$$

Usually, the new units are identical to the failed one when it was new. In the case of an n-fold standby redundancy, the statistical moments are given by [5]

$$\begin{aligned}
\mathsf{E}[T] &= n\mathsf{E}[T_1],\\
\mathsf{E}[T^2] &= n(n-1)\mathsf{E}[T_1]^2 + n\mathsf{E}[T_1^2],\\
\mathsf{E}[T^3] &= n(n-1)(n-2)\mathsf{E}[T_1]^3\\
&\quad + 3n(n-1)\mathsf{E}[T_1]E[T_1^2] + n\mathsf{E}[T_1^3],\\
\mathsf{E}[T^4] &= n(n-1)(n-2)(n-3)\mathsf{E}[T_1]^4\\
&\quad + 6n(n-1)(n-2)\mathsf{E}[T_1]^2\mathsf{E}[T_1^2]\\
&\quad + 4n(n-1)\mathsf{E}[T_1]\mathsf{E}[T_1^3]\\
&\quad + 3n(n-1)\mathsf{E}[T_1^2]^2 + n\mathsf{E}[T_1^4]
\end{aligned} \tag{7}$$

B. k-out-of-n Systems and Series Systems

With its four parameters, the Pearson method can model a wide range of distribution shapes, including standard distributions, such as the negative exponential, Gaussian, and uniform distribution, but also failure time distributions of real components/units. As the Pearson method, and its mathematics have been described in [3], [19], only the most salient features of its application are reported here. Specific issues arising in the numerical implementation are detailed in [15]. In the following, we shall use $g(x)$ to denote the Pearson distribution, to distinguish between this approximate pdf, and the actual failure time pdf of a constituent unit.

Pearson [3], [19] shows that the derivative of any pdf $g(x)$ can be approximated by

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = \frac{(x+a)g(x)}{d_0 + d_1 x + d_2 x^2} \tag{8}$$

For a *centralized* pdf (with zero first moment), the parameters a, d_0, d_1, d_2 of the Pearson approximation can be easily computed from the second to fourth moments $u_2, \ldots, u_4$ of $f_1(x)$ according to

$$\begin{aligned} a &= \left(3u_2^2 u_3 + u_3 u_4\right)/q \\ d_0 &= \left(-4u_2^2 u_4 + 3u_2 u_3^2\right)/q \\ d_1 &= \left(-3u_2^2 u_3 - u_3 u_4\right)/q \\ d_2 &= \left(3u_3^2 + 6u_2^3 - 2u_2 u_4\right)/q \end{aligned} \tag{9}$$

where $q = -18u_2^3 + 10u_2u_4 - 12u_3^2$. Using a centralized pdf significantly improves on numerical accuracy. Hence, to approximate a unit failure time pdf with arbitrary moments $m_1, \ldots, m_4$, we approximate a centralized version of the failure time pdf. Consequently, the moments $u_1, \ldots, u_4$ on which the centralized Pearson approximation $g(x)$ is based, is computed according to the translation

$$\begin{aligned} u_1 &= 0 \\ u_2 &= m_2 - m_1^2 \\ u_3 &= m_3 - 3m_1m_2 + 2m_1^3 \\ u_4 &= m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4 \end{aligned} \tag{10}$$

Depending on the roots of the equation

$$d_0 + d_1 x + d_2 x^2 = 0 \tag{11}$$

the analytical integration of (8) results in various analytical expressions for $g(x)$ [3], [19]. In the following, we present the derivation of $g(x)$ when the roots of (11) are real, and have different signs (coined Pearson pdf of type I [3], the other two cases, and associated Pearson types II & III are detailed in [15]). For Pearson type I, (8) is integrated to

$$g^{\mathrm{I}}(x) = K(a_1 + x)^A (a_2 - x)^B \tag{12}$$

where the superscript I denotes the Pearson type, and K is a normalization constant such that

$$\int_{-a_1}^{a_2} K(a_1 + x)^A (a_2 - x)^B \mathrm{d}x = 1$$

When used as approximation of $f_1(x)$ in composite (3), or $f_i(x)$ in composite (5), $g^{\mathrm{I}}(x)$ must be retranslated by a value m_1. For the centralized $g_i^{\mathrm{I}}(x)$ the parameters a_1, a_2, A, B are given by

$$\begin{aligned} a_1 &= \frac{d_1 + \sqrt{d_1^2 - 4d_0d_2}}{2d_2} - m_1 \\ a_2 &= \frac{-d_1 + \sqrt{d_1^2 - 4d_0d_2}}{2d_2} + m_1 \\ A &= \frac{a_1 - a}{d_2(a_1 + a_2)} \\ B &= \frac{a_2 + a}{d_2(a_1 + a_2)}, \end{aligned} \tag{13}$$

and (12) is only valid for $-a_1 < x < a_2$, outside this domain $g_i^{\mathrm{I}}(x) = 0$. Further implementation details are presented in Section IV.

In the case of a series system where (12) is used as an approximation of each $f_i(x)$ in the composite (5), each Pearson approximation $g_i(x)$ may represent a different mean value, and different computational bounds. Care must be taken to assure that in the numerical analysis the computation interval will include all $g_i(x)$. A different mean value will be accommodated by a change in the value of x in the functions concerned (for more details see Section IV-A-1).

1) Special Cases: In the Pearson approximation, we distinguish the following special cases.

- **Gaussian pdf.**
  In case we know beforehand that the pdf of $f_i(x)$ is of Gaussian type, the first four moments of the centralized density function $g_i(x)$ are known to be $u_1 = 0$, $u_2 = \sigma^2$, $u_3 = 0$, $u_4 = 3\sigma^4$, where σ^2 is the variance. In this case, the Pearson distribution type II must be applied. However, we must add an extremely small offset of $10^{-9}u_4$ to u_4 to have the Pearson approximation function correctly. The Pearson approximation of the exact Gaussian distribution does not work correctly, as its representation by point $B_1 = 0$, $B_2 = 3$ in Table 1 of [15] lies on the border of the region type II. By adding the mentioned offset, the Gaussian distribution is drawn into region II. However, the offset is so small that it does not affect the accuracy.
- **Negative exponential pdf.**
  When we know beforehand that the pdf of $f_i(x)$ is of the negative exponential type, the first four moments of the centralized density function $g_i(x)$ are known to be $u_1 = 0$, $u_2 = m_1^2$, $u_3 = 2m_1^3$, $u_4 = 9m_1^4$, where m_1 is the first moment of $f_1(x)$. In this case, the Type III Pearson approximation must be applied. Similar to the Gaussian case, we must add an extremely small offset of $10^{-9}u_4$ to u_4 in order to have the Pearson approximation function correctly. The negative exponential pdf lies on the border of

Pearson type III, and is represented by point $B_1 = 4$, $B_2 = 9$ in Table 1 of [15]. Note, of course, that for negative exponential distributions, it is possible to analytically derive exact solutions for k-out-of-n systems.

IV. Numerical Implementation

In this section, we describe the implementation of (3) (k-out-of-n systems), and (5) (series systems), which involve the use of Pearson distributions. Following Section III-B, we mainly present implementation details for Pearson type I pdf only, as the implementation details for the other Pearson types are similar, except for the expressions for $g_i(x)$, and their domains [15].

Regarding a practical implementation of (12), it turns out that for certain d_i solutions to (11) the factors A, and B in (12) can become rather large, which means that $(a_1 + x)^A$, and $(a_2 + x)^B$ can become extremely large, or extremely small. To avoid numerical precision errors, we have normalized both terms of (12) to

$$\frac{1}{c}\left[\frac{2(a_1+x)}{a_1+a_2}\right]^A \left[\frac{2(a_2-x)}{a_1+a_2}\right]^B = \frac{1}{c}\, g^{\mathrm{I}}(x) \qquad (14)$$

where

$$c = \int_{-a_1}^{a_2} g^{\mathrm{I}}(x)\,\mathrm{d}x \qquad (15)$$

For completeness, we also provide the equations corresponding to our implementation for the Pearson type II:

$$\frac{1}{c}\left[1+\left(\frac{x}{c}-\frac{v}{r}\right)^2\right]^{-m} e^{-v\arctan\left(\frac{x}{c}-\frac{v}{r}\right)} = \frac{1}{c}\, g^{\mathrm{II}}(x), \qquad (16)$$

and type III:

$$\frac{1}{c}\left(\frac{a_1+x}{a_1-a_2}\right)^A \left(\frac{2(a_2+x)}{a_2+15\sqrt{u_2}}\right)^B = \frac{1}{c}\, g^{\mathrm{III}}(x) \qquad (17)$$

where c is the corresponding normalization constant.

A. *k-out-of-n Systems*

To illustrate the computation process for k-out-of-n systems, Fig. 1 shows a simple algorithm to compute the resulting four moments of the composite pdf $f(x)$ ($\mathsf{E}[T^r]$, denoted M_r in the algorithm) from the moments of the constituent units ($\mathsf{E}[T_1^r]$, denoted m_r in the algorithm). In the algorithm, the variables f_1, F_1 reflect the fact that all units i are identical to unit $i = 1$. The computation of the resulting moments (algorithm lines 13 through 15) includes the translation from the centralized pdf domain back to the original domain.

To simplify the presentation, the algorithm is not optimized, and uses simple Euler integration (using integration step variable h). The complexity of the entire procedure is determined by the integration loop which involves approximately 50 floating point operations per integration step, independent of the system size n. For the accuracy needed (less than 1 percent approximation error), only 10^4 (Euler) integration steps are required (the loop bounds in algorithm lines 3, and 8), resulting in approximately 500,000 floating point operations for the computation of the four moments. This number can even be substantially reduced when more advanced integration methods are used. Thus, the algorithm pairs excellent accuracy at minimum computing cost. A C-implementation of the above (unoptimized) algorithm, compiled with gcc 2.95 (default optimization), only takes 0.040 CPU seconds on a (slow) 750 MHz AMD Duron processor running Linux Debian 2.2. The algorithm also applies to the other Pearson pdf types, in which case the computations in algorithm lines 1, 5, and 10, and the integration bounds on x are altered accordingly [15].

Algorithm 1:

function KOUTOFN($m_1, \ldots, m_4$) **returns** $M_1, \ldots, M_4$

1:	$a_1, a_2, A, B \leftarrow$ Eq. (13)	$g^{\mathrm{I}}(x)$
2:	$c \leftarrow 0$	
3:	**for** $h = 1, \ldots, 10^4$ **do**	
4:	$x \leftarrow -a_1 + h(a_1+a_2)/10^4$	
5:	$c \leftarrow c + g^{\mathrm{I}}(x)$	Eq. (15)
6:	**end for**	
7:	$F_1, M_1, \ldots, M_4 \leftarrow 0$	
8:	**for** $h = 1, \ldots, 10^4$ **do**	
9:	$x \leftarrow -a_1 + h(a_1+a_2)/10^4$	
10:	$f_1 \leftarrow g^{\mathrm{I}}(x)/c_1$	Eq. (14)
11:	$F_1 \leftarrow F_1 + f_1$	$F_1(x)$
12:	$f \leftarrow \frac{n! F_1^{n-k}(1-F_1)^{k-1} f_1}{(k-1)!(n-k)!}$	Eq. (3)
13:	$M_1 \leftarrow M_1 + (x+m_1)f$	$\mathsf{E}[T]$
14:	...	
15:	$M_4 \leftarrow M_4 + (x+m_1)^4 f$	$\mathsf{E}[T^4]$
16:	**end for**	

Fig. 1. Algorithm for k-out-of-n systems with Pearson type I pdf.

1) Series Systems: Essentially the same approach is used for series systems with different units. In that case however, the computations on algorithm lines 1, 2, 5, 10, and 11 need to be performed for each subsystem (unit) i, involving variables such as c_i, f_i, F_i. In addition, the computation on algorithm line 12 is altered according to (5). To cope with different mean values of $f_i(x)$, all pdf are collectively shifted along the x-axis such that the mean of the pdf with the smallest mean, $f_s(x)$ say, reaches zero. In the numerical analysis, the shifted functions must all be covered by one & the same computational (integration) interval, i.e., the lowest bound, and the highest bound of all pdf will be taken as the computational bounds. Suppose that after the shift to centralize $g_s(x)$ the other functions $g_i(x), i \neq s$ obtain mean values φ_i, $i \neq s$. As usual, the Pearson parameters of the functions $g_i(x), i \neq s$ are derived from *their* centralized versions, but, in addition, and before applying (5) to derive the composite density function $f(x)$, x must now be replaced by $x - \varphi_i$ ($\varphi_s = 0$) in the functions $g_i(x)$. Finally, the moments of $f(x)$ are obtained in the same manner as given in lines 13–15 of Algorithm 1.

V. Example Applications

In this section, we apply our analysis approach to a number of systems from the aerospace domain, which involves compositions of non-identical units, standby redundancy, and active redundant k-out-of-n compositions of identical subsystems (hierarchical composition). While in [15] we have provided ample proof of the basic accuracy of our method, in the following we also provide error values, which also show the high accuracy for hierarchical composition. To evaluate the accuracy of our approach, we compare our approximations with analytic, and pseudo-analytic solutions, the procedure of which is also described.

A. Series System of Different Units

As an example, we present the failure time calculation of a satellite transponder, which relays signals received from Earth back to Earth. Such a transponder has a number of semi-conductor components, and (in older versions) an output traveling wave tube (TWT). The set of all semi-conductor elements together is assumed to have a negative exponential failure time pdf with an assumed mean of 7 years. The TWT is assumed to have a Gaussian failure time pdf with a mean of 7 years, and a variance of 2 year^2. The centralized Gaussian pdf is approximated by a Pearson type II pdf with zero mean, second moment of 2 year^2, zero third moment, and a fourth moment of 12.00000005 year^4 (see also Section III-B-1). The centralized, negative exponential pdf is approximated using a Pearson type III pdf with a zero mean, second moment of 49 year^2, third moment of 686 year^3, and a fourth moment of 21609.00001 year^4 (see also Section III-B-1).

Applying our approximation method for the binary series system with different units (i.e., (5) with $n = 2$ with $g_i(x)$ according to Pearson type II, and III, respectively), we obtain the following four failure time moments: $\mathsf{E}[T] = 4.3716$ years (error: 0.01 percent), $\mathsf{E}[T^2] = 25.91\ \mathrm{year}^2$ (error: 0.01 percent), $\mathsf{E}[T^3] = 172.8\ \mathrm{year}^3$ (error: 0.01 percent), and $\mathsf{E}[T^4] = 1,235.3\ \mathrm{year}^4$ (error: 0.01 percent).

To obtain the error values of the above four moments, we have derived two additional solutions to (5) for $n = 2$, i.e., an analytic solution, and a pseudo-analytic one. The derivation of the first moment of the analytic solution is shown in the Appendix. Like the analytic solution, the pseudo-analytic solution is based on the use of the negative exponential pdf $(f_1(x))$, and the Gaussian pdf $(f_2(x))$ instead of the Pearson approximations. Unlike the analytic solution, however, the Gaussian cdf is numerically determined. The computation is similar to the version of Algorithm 1 described in Section IV-A-1. Within the integration loop (algorithm lines 8 to 16), the variables f_1 (negative exponential pdf), f_2 (Gaussian pdf), F_1 (negative exponential cdf, equals $1 - f_1/\lambda$), and f (composite pdf) are computed analytically in each step, while F_2 (Gaussian cdf) is computed through Euler integration.

The results of the first moment for the analytic, pseudo-analytic, and our Pearson approximation are 4.3722, 4.3713, and 4.3716, respectively. We conclude that the pseudo-analytic method provides results which are extremely close to those of the analytical method, and for that reason can be considered to provide the true moment values. Hence, this numerical integration method will also be used as error reference in the following sections.

B. Hierarchy of k-out-of-n Systems

To improve the mission time of the transponder of Section V-A, we consider the addition of one redundant, identical transponder. To give this redundant switched-on transponder some useful task during the period that the primary transponder functions correctly, the redundant transponder is assigned a secondary mission for this time period. To avoid the use of high frequency switches on-board, the secondary mission is proposed to operate in a slightly higher frequency band than the primary mission. Each of the two transponders should then be equipped with a filter; the primary one with a low-frequency pass filter, and the redundant one with a high-frequency pass filter. When the primary transponder fails, the secondary mission will be terminated by stopping the related ground station transmissions, and the primary mission continues on the redundant transponder using the frequency band of the secondary mission. When the redundant transponder fails, the secondary mission is terminated as well. Two failure time distributions can be considered. The first one deals with the failure time of the primary mission, and the second one with that of the secondary mission.

Applying our approximation method for 1-out-of-2 systems using the moments obtained in Section V-A (i.e., a hierarchical system composition, using Algorithm 1 with $g(x)$ according to Pearson type I, and (3) on algorithm line 12 with $k = 1$, $n = 2$), the following failure time moments of the primary mission are found: $\mathsf{E}[T] = 5.876$ year (error: 0.05 percent), $\mathsf{E}[T^2] = 39.09\ \mathrm{year}^2$ (error: 0.14 percent), $\mathsf{E}[T^3] = 278.6\ \mathrm{year}^3$ (error: 0.26 percent), and $\mathsf{E}[T^4] = 2,076.0\ \mathrm{year}^4$ (error: 0.10 percent).

Applying our approximation method for 2-out-of-2 systems on the moments obtained for the series system in Section V-A (again, a hierarchical system composition, using Algorithm 1 with $g_i(x)$ according to Pearson type I, and (3) on algorithm line 12 with $k = 2$, $n = 2$), the following failure time moments of the secondary mission are obtained: $\mathsf{E}[T] = 2.867$ year (error: 0.09 percent), $\mathsf{E}[T^2] = 12.72\ \mathrm{year}^2$ (error: 0.41 percent), $\mathsf{E}[T^3] = 67.16\ \mathrm{year}^3$ (error: 1.08 percent), and $\mathsf{E}[T^4] = 394.38\ \mathrm{year}^4$ (error: 0.57 percent).

To derive the above errors in the moments of both the primary, and secondary mission, we use a pseudo-analytical approach. The approach is based on reusing Algorithm 1, which has previously been used for the computation of the results of the series subsystem in Section V-A. These results are now input to the current computation. Instead of separating the computations of Sections V-A, and V-B, which would require a Pearson approximation for the resulting moments of Section V-A, we simply merge the computations of Sections V-A, and V-B within a single integration loop in terms of Algorithm 1. The results of the integration loop for $f^{(\mathrm{series})}$ (using (5)) in terms of the series subsystem of Section V-A, are directly input to the computation for $f^{(k-\mathrm{out-of}-n)}$ (using (3)). An additional variable $F^{(\mathrm{series})}$ is used to hold the cdf of the series subsystem. By avoiding the Pearson approximation entirely, combined with the high accuracy of the integration loop (as previously shown), the pseudo-analytical solution has negligible error. The latter has been verified through a fully analytical solution for $\mathsf{E}[T]$ using a series expansion for $f_1(x)$ (not presented).

C. Standby Redundancy

With standby redundancy, it is assumed that the failing unit is replaced by an identical, new spare. Then the four moments of the composite system can directly be expressed in the four moments of the first unit according to (7). When applying standby redundancy on the example of Section V-A, one obtains the following results: $\mathsf{E}[T] = 8.74$ year (error: 0.01 percent), $\mathsf{E}[T^2] = 90.04$ year^2 (error: 0.01 percent), $\mathsf{E}[T^3] = 1025.30$ year^3 (error: 0.01 percent), and $\mathsf{E}[T^4] = 12,541.87$ year^4 (error: 0.02 percent). As expected, standby redundancy provides a better mean time reliability than active redundancy. When having standby redundancy, a secondary mission is not possible (like in Section V-B). On the other hand, we save power for the satellite.

D. Hierarchy With Triple Modular Redundancy

A triple modular redundancy (TMR) system is operational as long as 2 or all 3 units function correctly. In real life, and for commercial reasons, it is not realistic to apply TMR to the satellite transponder unit of Section V-A. Nevertheless we have taken the derived transponder moments of Section V-A as input to a higher, hierarchical TMR system, again to demonstrate its proper functioning when non-standard pdf are used as input. Clearly, however, we could have taken a more realistic example such as a gyro system, which would also be modeled as a series system, comprised of electronics, and a mechanical unit. Using the derived moments in Section V-A as input to the k-out-of-n system with $k = 2$, and $n = 3$, one obtains the following failure time moments: $\mathsf{E}[X] = 4.3692$ year (error: 0.0735 percent), $\mathsf{E}[X^2] = 22.9767$ year^2 (error: 0.705 percent), $\mathsf{E}[X^3] = 134.3904$ year^3 (error: 0.649 percent), and $\mathsf{E}[X^4] = 844.648$ year^4 (error: 0.337 percent). Again, the approximation errors are quite small. These error values have been obtained in the same way as described in Section V-B using a very similar peudo-analytic method.

VI. Conclusion

In this paper, we present a low-cost, compositional approach based on the use of the first four statistical moments to characterize the (unimodal) failure time distributions of the constituent subsystems, and the composite system. The approach is based on the use of Pearson distributions as an intermediate analytical vehicle, in terms of which the constituent failure time distributions are approximated. The analysis technique has been presented for k-out-of-n systems with identical subsystems, series systems with different subsystems, and systems exploiting standby redundancy. In addition, numeric implementation details have been outlined, and a number of example applications from the aerospace domain have been presented. The technique consistently exhibits very good accuracy (on average, much less than 1 percent error) at very modest computing cost, independent of system size.

Future work is aimed at extending the analysis for *dependent* subsystems, based on results from order statistics when variates exhibit a non-zero covariance. Typically, accurate analysis results are available for Gaussian distributions only. However, using Johnson distributions, one can transform arbitrary pdf into Gaussian pdf, subsequently applying our approach. Another direction of future research is the use of BDD, extending the analysis to higher moments. Possibly the work presented in [4] can be combined with our method to approximate a pdf by means of the Pearson method. After a Boolean evaluation of the failure time function as illustrated in [16], the remaining distribution functions can be approximated using the Pearson method. This would extend the work in [16], which is still limited to exponential polynomial distributions.

Appendix
Analytical Solution for Section V-A

We analytically derive the first moment $\mathsf{E}[T]$ of the failure time pdf of a series system with a Gaussian, and a negative exponential pdf. For $n = 2$, (5) reduces to

$$f(x) = (1 - F_1(x))\, f_2(x) + (1 - F_2(x))\, f_1(x)$$

where $f_1(x)$, and $f_2(x)$ represent a negative exponential pdf $(1/\lambda = 7)$, and a Gaussian pdf $(\mu = 7)$, respectively.

By translating $f_i(x)$ to a zero first moment (a backward shift of $1/\lambda = 7$), we obtain

$$f_1(x) = \frac{\lambda e^{-\lambda x}}{e}, \quad F_1(x) = 1 - \frac{f_1(x)}{\lambda}$$

and (for $\sigma = \sqrt{2}$)

$$f_2(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}, \quad F_2(x) = -\frac{2}{x}\,\frac{f_2(x)}{x}$$

The associated domains of $f_1(x)$, and $f_2(x)$ are $-1/\lambda < x < \infty$, and $-\infty < x < \infty$, respectively. The derivation proceeds as shown at the top of the next page. Using a series expansion for $f_1(x)$, it follows that

$$\begin{aligned}
\mathsf{E}[T] &= -\frac{1}{\lambda^2}\frac{\lambda}{e}\left[\int_{-1/\lambda}^{+\infty} f_2(x)\mathrm{d}x - \lambda \int_{-1/\lambda}^{+\infty} x f_2(x)\mathrm{d}x\right] \\
&\quad + \frac{1}{\lambda^2}\frac{\lambda}{e}\left[\frac{\lambda^2}{2}\int_{-1/\lambda}^{+\infty} x^2 f_2(x)\mathrm{d}x - \frac{\lambda^3}{6}\int_{-1/\lambda}^{+\infty} x^3 f_2(x)\mathrm{d}x\right] \\
&= -\frac{1}{\lambda e}\left[1 - 0 + \frac{\lambda^2}{2}\int_{-1/\lambda}^{+\infty} x^2 f_2(x)\mathrm{d}x - 0\right] \\
&= -\frac{1}{\lambda e}\left[1 + \frac{2\lambda^2}{2}\int_{-1/\lambda}^{+\infty} f_2(x)\mathrm{d}x\right] \\
&= -\frac{1}{\lambda e}\left[1 + \lambda^2\right]
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}[T] &= \int_{-\infty}^{+\infty} x f_2 \mathrm{d}x - \int_{-1/\lambda}^{+\infty} x F_1 f_2 \mathrm{d}x + \int_{-\infty}^{+\infty} x f_1 \mathrm{d}x - \int_{-1/\lambda}^{+\infty} x F_2 f_1 \mathrm{d}x \\
&= - \int_{-1/\lambda}^{+\infty} x F_1 f_2 \mathrm{d}x - \int_{-1/\lambda}^{+\infty} x F_2 f_1 \mathrm{d}x \\
&= - \int_{-1/\lambda}^{+\infty} x \left(1 - \frac{f_1(x)}{\lambda}\right) f_2 \mathrm{d}x - \int_{-1/\lambda}^{+\infty} x F_2 f_1 \mathrm{d}x \\
&= \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} x f_1 f_2 \mathrm{d}x - \int_{-1/\lambda}^{+\infty} x F_2 f_1 \mathrm{d}x \\
&= \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} x f_1 f_2 \mathrm{d}x + \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} x F_2 \mathrm{d} f_1 \\
&= \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} x f_1 f_2 \mathrm{d}x - \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} f_1 \mathrm{d}(x F_2) \\
&= \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} x f_1 f_2 \mathrm{d}x - \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} f_1 f_2 x \mathrm{d}x - \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} f_1 F_2 \mathrm{d}x \\
&= - \frac{1}{\lambda} \int_{-1/\lambda}^{+\infty} f_1 F_2 \mathrm{d}x \\
&= \frac{1}{\lambda^2} \int_{-1/\lambda}^{+\infty} F_2 \mathrm{d} f_1 \\
&= - \frac{1}{\lambda^2} \int_{-1/\lambda}^{+\infty} f_1 f_2 \mathrm{d}x
\end{aligned}$$

as $1/\lambda$ is sufficiently larger than σ. For $1/\lambda = 7$, it follows that $\mathsf{E}[T] = -2.6278$. Retranslating the result by a forward shift of the first moment $(1/\lambda = 7)$ yields $\mathsf{E}[T] = 4.3722$.

Acknowledgment

The authors would like to thank the anonymous referees and the associate editor Dr. M. Xie for their valuable comments, which greatly improved the paper.

References

[1] M. Asadi and I. Bayramoglu, "The mean residual life function of a k-out-of- n structure at the system level," *IEEE Trans. Reliab.*, vol. 55, pp. 314–318, 2006.

[2] Y. Chen and Q. Yang, "Reliability of two-stage weighted- k-out-of- n systems with components in common," *IEEE Trans. Reliab.*, vol. 54, pp. 431–440, 2005.

[3] W. P. Elderton and N. L. Johnson, *Systems of Frequency Curves*. : Cambridge University Press, 1969, pp. 35–81.

[4] H. Gautama and A. J. C. van Gemund, "Reliability analysis of k-out-of- n systems using generalized lambda distributions," in *Proc. of the Int'l Conference on Mathematics and Its Applications (SEAM—GMU 2003)*, Yogyakarta, Indonesia, 2003, pp. 466–474.

[5] H. Gautama, "A statistical approach to the performance modeling of parallel systems," Ph.D. dissertation, Delft University of Technology, , The Netherlands, 2004.

[6] H. Gautama and A. J. C. van Gemund, "Low-cost static performance prediction of parallel stochastic task compositions," *IEEE Trans. Par. Distr. Syst.*, vol. 17, pp. 78–91, 2006.

[7] E. Gelenbe, E. Montagne, and R. Suros, "A performance model of block structured parallel programs," in *Parallel Algorithms and Architectures*, M. Cosnard, Ed. *et al.* : Elsevier Science Publishers, North-Holland, 1986, pp. 127–138, Cosnard, M. et al.(eds).

[8] E. J. Gumbel, "Statistical theory of extreme values (main results)," in *Contributions to Order Statistics*, A. E. Sarhan and B. G. Greenberg, Eds. : John Wiley & Sons, 1962, pp. 56–93.

[9] C. P. Kruskal and A. Weiss, "Allocating independent subtasks on parallel processors," in *Proc. Int'l Conf. Parallel Proc.*, 1986, pp. 145–152, IEEE.

[10] X. Li and M. J. Zuo, "On the behavior of some new aging properties based upon the residual life of k-out-of- n systems," *J. Applied Probability*, vol. 39, pp. 426–433, 2002.

[11] D.-R. Liang and S. K. Tripathi, "On performance prediction of parallel computations with precedence constraints," *IEEE Trans. Par. Distr. Syst.*, vol. 11, pp. 491–508, 2000.

[12] S. Madala and J. Sinclair, "Performance of synchronous parallel algorithms with regular structures," *IEEE Trans. Par. Distr. Syst.*, vol. 2, pp. 105–116, 1991.

[13] R. H. Myers, K. L. Wang, and H. M. Gurdy, *Reliability Engineering for Electronic Systems*. : Wiley, 1964.
[14] J. S. Ramberg, P. R. Tadikamalla, E. J. Dudewicz, and F. M. Mykytka, "A probability distribution and its uses in fitting data," *Technometrics*, vol. 21, pp. 201–214, 1979.
[15] G. L. Reijns and A. J. C. van Gemund, "Predicting execution times of parallel-independent programs using Pearson distributions," *Parallel Computing*, vol. 31, pp. 877–899, 2005.
[16] R. A. Sahner and K. S. Trivedi, "Performance and reliability analysis using directed acyclic graphs," *IEEE Trans. Softw. Eng'g*, vol. SE-13, pp. 1105–1114, 1987.
[17] A. Sarhan, "Reliability equivalence factors of a parallel system," *Reliability Engineering and System Safety*, vol. 87, pp. 405–411, 2005.
[18] D. P. Siewiorek and R. S. Swarz, *Reliable Computer Systems, Design and Evolution*, 2nd ed. : Digital Press, 1982.
[19] A. Stuart and J. K. Ord, *Kendall's Advanced Theory of Statistics*, 6th ed. New York: Halsted Press, 1994, vol. 1.
[20] Z. Tang and J. B. Dugan, "BDD-based reliability analysis of phased-mission systems with multimode failures," *IEEE Trans. Reliab.*, vol. 55, pp. 350–360, 2006.
[21] Z. Tian and M. J. Zuo, "Redundancy allocation for multi-state systems using physical programming and genetic algorithms," *Reliability Engineering and System Safety*, vol. 91, pp. 1049–1056, 2006.
[22] K. S. Trivedi, *Probability and Statistics with Reliability, Queuing and Computer Science Applications*. : Prentice Hall, 1982.
[23] Z. Zang, D. Wang, H. Sun, and K. S. Trivedi, "A BDD-based algorithm for analysis of multistate systems with multistate components," *IEEE Trans. Comput.*, vol. 52, pp. 1608–1618, 2003.

Gerard Reijns received a MSc. in Electrical Engineering from Delft University of Technology. He worked at the PTT laboratory, an air defense lab, and at the European Space Agency in the period 1956-1972. He joined, as an Associate Professor, Delft University of Technology in 1972, and became Full Professor in 1980. At present he is an emeritus professor, but still active two days per week at the Faculty of Electrical Engineering, Mathematics, and Computer Science; and at the Faculty of Aerospace Engineering. His fields of special interest are computer performance, parallel processing, and image archiving and communication systems for hospitals. He was consultant to the Netherlands Aerospace Agency from 1971 to 1995, and chairman of Technical Committee 10 of the International Federation for Information processing from 1987 to 1993. He is author of more than 100 articles, and co-editor of several books.

Arjan J. C. van Gemund received a BS degree in Physics in 1981, an MS degree cum laude in Computer Science in 1989, and a PhD degree cum laude in 1996, all from Delft University of Technology. Between 1981 and 1985, he worked as an Electrical Engineer, and Systems Programmer at the R & D organization of a Dutch multinational company. Between 1989 and 1992, he joined the Dutch TNO research organization as a Research Scientist specialized in the field of high-performance computing. Since 1992, he is a faculty member at the Faculty of Electrical Engineering, Mathematics, and Computer Science of Delft University of Technology, currently serving as a Full Professor. His research interests are in the area of systems fault diagnosis, systems dependability, and parallel and distributed systems programming. He (co)authored more than 100 scientific papers.